Richmond H. Thomason

Symbolic Logic

An Introduction

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I Uninterpreted Syntax of a Logical System

1. This section has to do with some rules for constructing certain strings of symbols called *formulas*, and for manipulating these formulas to build arrays called *proofs*.

Some of you have studied formulas and proofs of this sort before; for the present, please try to forget what you have learned about them. The idea is to approach this material without any reference to what the "formulas" and "proofs" may mean; the rules for manipulating symbols are enough like the rules of games so that they can be regarded in a gamelike way, without regard to anything external.

Incidentally, this sort of "horseshoe pushing" is one way of caricaturing what logicians do. But despite this, many logicians aren't very interested in this sort of thing; what they do is much more abstract and conceptual. As we go on to more advanced topics you will be able to see the contrast between these two sorts of activities.

2. All our formulas will be strings made up of the following eight symbols.

$$\begin{array}{cccc} p & q & r & s \\ (&) & \supset & \sim \end{array}$$

For instance, the following are formulas.

$$\begin{array}{ll} (p \supset q) & ((p \supset q) \supset (r \supset s)) \\ \sim r & s \\ \sim(\sim(r \supset s) \supset \sim\sim p) & \sim\sim\sim(q \supset q) \\ p & ((p \supset q) \supset q) \end{array}$$

But not every string of symbols is a formula; for instance, the following are not formulas.

$$\begin{array}{ll} \sim(\sim p) &)p(\\ p \supset (q) & ((p \supset \sim q) \supset \sim q \\ p \supset q & \supset \end{array}$$

These examples should give you an idea of how to recognize and make formulas; the idea is that any of the four letters 'p', 'q', 'r', and 's' is a formula, and that more complicated formulas can be made by applying the symbol '$\sim$' to the left of any formula, or the symbol '$\supset$' between any two formulas. In the latter case, notice that parentheses have to be added in the appropriate places.

Note: Here are some things to think about. How long can formulas get? How many formulas are there? What is the difference between saying there is no bound on the length of formulas and saying that there is no formula of unbounded length? These are questions that will seem familiar to those with some mathematical training, but others may have difficulty with them at first. In the latter case, you should do some thinking about these matters, and may want to read something about elementary set theory. See Chapter XIII of this book, or one of the works listed in the bibliography.

3. *Proofs* are built up from *axioms* using certain *rules of inference*. The directions for generating proofs are quite simple: any axiom may be written down at any time in a proof, and rules may be applied to any formulas already written down in a proof. Often we are interested in finding proofs *of* certain formulas; a proof is said to be a proof of the formula that is its last item.

The system S_0 with which we will be dealing for the time being has just three axioms; these are the formulas

A1. $(p \supset (q \supset p))$

A2. $((p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r)))$

A3. $((\sim p \supset \sim q) \supset (q \supset p))$.

The following columns of formulas qualify as proofs in S_0.

$$(p \supset (q \supset p))$$

(*i*)

$$(p \supset (q \supset p))$$
$$(p \supset (r \supset p))$$
$$(r \supset (r \supset r))$$
$$((\sim p \supset \sim q) \supset (q \supset p))$$

(*ii*)

$$(p \supset (q \supset p))$$
$$(p \supset (p \supset p))$$
$$((p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r)))$$
$$((p \supset (p \supset r)) \supset ((p \supset p) \supset (p \supset r)))$$
$$((p \supset (p \supset p)) \supset ((p \supset p) \supset (p \supset p)))$$
$$((p \supset p) \supset (p \supset p))$$

(*iii*)

$$((\sim p \supset \sim q) \supset (q \supset p))$$
$$((\sim\sim p \supset \sim q) \supset (q \supset \sim p))$$
$$((\sim\sim p \supset \sim\sim\sim p) \supset (\sim\sim p \supset \sim p))$$

(*iv*)

$$((p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r)))$$
$$(((\sim p \supset \sim\sim p) \supset (q \supset r)) \supset (((\sim p \supset \sim\sim p) \supset q) \supset ((\sim p \supset \sim\sim p) \supset r)))$$
$$(((\sim p \supset \sim\sim p) \supset (\sim p \supset r)) \supset (((\sim p \supset \sim\sim p) \supset \sim p) \supset ((\sim p \supset \sim\sim p) \supset r)))$$
$$(((\sim p \supset \sim\sim p) \supset (\sim p \supset p)) \supset (((\sim p \supset \sim\sim p) \supset \sim p) \supset ((\sim p \supset \sim\sim p) \supset p)))$$
$$((\sim p \supset \sim q) \supset (q \supset p))$$
$$((\sim p \supset \sim\sim p) \supset (\sim p \supset p))$$
$$(((\sim p \supset \sim\sim p) \supset \sim p) \supset ((\sim p \supset \sim\sim p) \supset p))$$

(*v*)

Before reading the explanation given below of the rules of proof construction, you may wish to work them out for yourself by inspection of the above examples.

4. There are just two rules of inference in the system S_0. The first, *modus ponens*, may be used in case you have already gotten two steps in a proof, one of which is the result of putting the symbol '⊃' after the other, following the '⊃' by a formula, and then enclosing the whole in parentheses. *Modus ponens* then allows you to infer the second of these formulas.

Notice how obscure and complicated the above explanation has become. It is much easier to say things of this sort by making a picture. We can diagram the rule *modus ponens* as follows.

$$\frac{\text{---} \qquad (\text{---} \supset \ldots)}{\ldots}$$

6. The second rule of S_0 is a rule of substitution; the rule allows you to replace 'p', 'q', 'r', or 's' *in* any formula *by* any formula. One must replace *all* occurrences of the letter substituted for in the formula on which one is working; the following is not an instance of the substitution rule.

$$\frac{(p \supset (q \supset p))}{(p \supset (q \supset (p \supset p)))}$$

Although '(p ⊃ p)' is substituted for the second occurrence of 'p' in '(p ⊃ (q ⊃ p))', it is not substituted for the first occurrence of 'p' in that formula.

We might express this aspect of the rule by saying that it is a rule of *simultaneous* substitution. But the substitutions must be done one at a time; the following also is not an instance of the substitution rule.

$$\frac{(p \supset (q \supset p))}{((p \supset p) \supset (r \supset (p \supset p)))}$$

We *can* get from '(p ⊃ (q ⊃ p))' to '((p ⊃ p) ⊃ (r ⊃ (p ⊃ p)))', but it takes two steps, as in the following proof.

$$\begin{array}{l} (p \supset (q \supset p)) \\ (p \supset (r \supset p)) \\ ((p \supset p) \supset (r \supset (p \supset p))) \end{array}$$

Notice that the substitution rule does not allow substitutions for formulas other than 'p', 'q', 'r', and 's', so that inferences like the following one also are not sanctioned by this rule.

$$\frac{(p \supset ((q \supset r) \supset p))}{(p \supset (q \supset p))}$$

The rule of substitution is quite a different thing from *modus ponens*, and cannot be diagrammed in the same way. We could invent special notation for displaying substitutions, but at present would gain nothing by doing so; the rule should now be clear enough so that you can recognize instances of it, and that is enough.

7. We can now characterize explicitly the notion of a proof in the system S_0; a proof is any column of formulas (the *steps* of the proof), such that every step of the column is an axiom, or follows from two previous steps of the column by means of *modus ponens*, or follows from one previous step of the column by means of substitution.

According to this definition a column which consists of just one step, that step being an axiom, is a proof. Thus every axiom of S_0 has a proof in S_0 (or is *provable* in S_0 or a *theorem* of S_0).

To help in recognizing proofs, it is convenient to number steps and to include justifications, as in the following annotated version of example iii.

1	$(p \supset (q \supset p))$	A1
2	$(p \supset (p \supset p))$	1, subst
3	$((p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r)))$	A2
4	$((p \supset (p \supset r)) \supset ((p \supset p) \supset (p \supset r)))$	3, subst
5	$((p \supset (p \supset p)) \supset ((p \supset p) \supset (p \supset p)))$	4, subst
6	$((p \supset p) \supset (p \supset p))$	2, 5, m p

(*iii′*)

Example iii′ is more than a proof; it is a proof together with auxiliary notations that make it easier to perceive as a proof.

Exercises

1. Which of the following are formulas of S_0?

   (a) r (b) $\sim\sim(p)$
   (c) $(p \supset (q \supset t))$ (d) $(q \supset (r \supset)s)$
   (e) $\sim\sim p \supset (q \supset (r \supset p))$ (f) $(((p \supset \sim\sim q) \supset \sim\sim\sim s) \supset \sim(p \supset p))$
   (g) $((((p \supset p) \supset p) \supset p) \supset p)$ (h) $(p \supset (p \supset (p \supset p)))$

2. Write down some formulas of S_0.
3. Annotate the proofs given in examples i, ii, iii, iv, and v.
4. Write down some proofs in S_0.
5. Find proofs in S_0 for the following formulas, and annotate these proofs.

   (a) $(p \supset (r \supset (r \supset r)))$
   (b) $(((\sim\sim p \supset \sim q) \supset q) \supset ((\sim\sim p \supset \sim q) \supset \sim p))$
   (c) $(p \supset p)$
   (d) $(\sim q \supset (q \supset p))$
   (e) $((p \supset q) \supset ((r \supset p) \supset (r \supset q)))$

Problems

1. Find a definition of 'formula of S_0'. Make it as rigorous as possible. (*Hint:* Consult the definition given in V.3, if necessary.)
2. Use your definition to show that if a string S of symbols follows from another string T of symbols by the rule of substitution, then S is a formula of S_0 if T is.
3. Try to construct a definition of 'theorem of S_0' like your definition of 'formula of S_0'. This definition should not involve the notion of a proof explicitly.

Solutions to Selected Exercises

Chapter I

1. The following are formulas of S_0: (a), (f), (g), (h).

5.	(a)	1	$(p \supset (q \supset p))$	A1
		2	$(r \supset (q \supset r))$	1, subst
		3	$(r \supset (r \supset r))$	2, subst
		4	$((r \supset (r \supset r)) \supset (q \supset (r \supset (r \supset r))))$	1, subst
		5	$(q \supset (r \supset (r \supset r)))$	3, 4, m p
		6	$(p \supset (r \supset (r \supset r)))$	5, subst
	(b)	1	$((\sim p \supset \sim q) \supset (q \supset p))$	A3
		2	$((\sim\sim p \supset \sim q) \supset (q \supset \sim p))$	1, subst
		3	$((p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r)))$	A2
		4	$(((\sim\sim p \supset \sim q) \supset (q \supset r)) \supset (((\sim\sim p \supset \sim q) \supset q)$ $\supset ((\sim\sim p \supset \sim q) \supset r)))$	3, subst
		5	$(((\sim\sim p \supset \sim q) \supset (q \supset \sim p))$ $\supset (((\sim\sim p \supset \sim q) \supset q) \supset ((\sim\sim p \supset \sim q) \supset \sim p)))$	4, subst
		6	$(((\sim\sim p \supset \sim q) \supset q) \supset ((\sim\sim p \supset \sim q) \supset \sim p))$	2, 5, m p

(c)	1	$((p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r)))$	A2
	2	$((p \supset ((p \supset p) \supset r)) \supset ((p \supset (p \supset p)) \supset (p \supset r)))$	1, subst
	3	$((p \supset ((p \supset p) \supset p)) \supset ((p \supset (p \supset p)) \supset (p \supset p)))$	2, subst
	4	$(p \supset (q \supset p))$	A1
	5	$(p \supset ((p \supset p) \supset p))$	4, subst
	6	$((p \supset (p \supset p)) \supset (p \supset p))$	3, 5, m p
	7	$(p \supset (p \supset p))$	4, subst
	8	$(p \supset p)$	6, 7, m p

(d)	1	$((\sim p \supset \sim q) \supset (q \supset p))$	A3
	2	$(p \supset (q \supset p))$	A1
	3	$(p \supset (\sim q \supset p))$	2, subst
	4	$(((\sim p \supset \sim q) \supset (q \supset p))$ $\supset (\sim q \supset ((\sim p \supset \sim q) \supset (q \supset p))))$	3, subst
	5	$(\sim q \supset ((\sim p \supset \sim q) \supset (q \supset p)))$	1, 4, m p
	6	$((p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r)))$	A2
	7	$((s \supset (q \supset r)) \supset ((s \supset q) \supset (s \supset r)))$	6, subst
	8	$((s \supset ((\sim p \supset \sim q) \supset r))$ $\supset ((s \supset (\sim p \supset \sim q)) \supset (s \supset r)))$	7, subst
	9	$((\sim q \supset ((\sim p \supset \sim q) \supset r))$ $\supset ((\sim q \supset (\sim p \supset \sim q)) \supset (\sim q \supset r)))$	8, subst
	10	$((\sim q \supset ((\sim p \supset \sim q) \supset (q \supset p)))$ $\supset ((\sim q \supset (\sim p \supset \sim q)) \supset (\sim q \supset (q \supset p))))$	9, subst
	11	$((\sim q \supset (\sim p \supset \sim q)) \supset (\sim q \supset (q \supset p)))$	5, 10, m p
	12	$(p \supset (r \supset p))$	2, subst
	13	$(\sim q \supset (r \supset \sim q))$	12, subst
	14	$(\sim q \supset (\sim p \supset \sim q))$	13, subst
	15	$(\sim q \supset (q \supset p))$	11, 14, m p

(e)	1	$((p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r)))$	A2
	2	$((p \supset (s \supset r)) \supset ((p \supset s) \supset (p \supset r)))$	1, subst
	3	$((q \supset (s \supset r)) \supset ((q \supset s) \supset (q \supset r)))$	2, subst
	4	$(p \supset (q \supset p))$	A1
	5	$(p \supset ((s \supset r) \supset p))$	4, subst
	6	$(((q \supset (s \supset r)) \supset ((q \supset s) \supset (q \supset r))) \supset ((s \supset r)$ $\supset ((q \supset (s \supset r)) \supset ((q \supset s) \supset (q \supset r)))))$	5, subst
	7	$((s \supset r) \supset ((q \supset (s \supset r)) \supset ((q \supset s) \supset (q \supset r))))$	3, 6, m p
	8	$((s \supset (q \supset r)) \supset ((s \supset q) \supset (s \supset r)))$	1, subst
	9	$((s \supset (q \supset p)) \supset ((s \supset q) \supset (s \supset p)))$	8, subst
	10	$(((s \supset r) \supset (q \supset p)) \supset (((s \supset r) \supset q)$ $\supset ((s \supset r) \supset p)))$	9, subst
	11	$(((s \supset r) \supset ((q \supset (s \supset r)) \supset p)) \supset (((s \supset r)$ $\supset (q \supset (s \supset r))) \supset ((s \supset r) \supset p)))$	10, subst
	12	$(((s \supset r) \supset ((q \supset (s \supset r)) \supset ((q \supset s) \supset (q \supset r))))$ $\supset (((s \supset r) \supset (q \supset (s \supset r))) \supset ((s \supset r)$ $\supset ((q \supset s) \supset (q \supset r)))))$	11, subst

13	(((s ⊃ r) ⊃ (q ⊃ (s ⊃ r))) ⊃ ((s ⊃ r) ⊃ ((q ⊃ s) ⊃ (q ⊃ r))))	7, 12 m p
14	((s ⊃ r) ⊃ (q ⊃ (s ⊃ r)))	4, subst
15	((s ⊃ r) ⊃ ((q ⊃ s) ⊃ (q ⊃ r)))	13, 14, m p
16	((p ⊃ r) ⊃ ((q ⊃ p) ⊃ (q ⊃ r)))	15, subst
17	((p ⊃ s) ⊃ ((q ⊃ p) ⊃ (q ⊃ s)))	16, subst
18	((p ⊃ s) ⊃ ((r ⊃ p) ⊃ (r ⊃ s)))	17, subst
19	((p ⊃ q) ⊃ ((r ⊃ p) ⊃ (r ⊃ q)))	18, subst

Chapter II

1. (a) '(~q ⊃ p)', where 'p' stands for 'We'll go to the beach today' and 'q' for 'It rains'.
   (b) '(p ⊃ ~q)', where 'p' stands for 'You do what you're told' and 'q' for 'You'll get along badly here'.
   (c) '~q', where 'q' stands for 'The dog was treated unkindly'.
   (d) '(p ⊃ (q ⊃ (~r ⊃ s)))', where 'p' stands for 'I move my pawn', 'q' for 'He castles', 'r' for 'I lose my queen', and 's' for 'I should be able to beat him'.
   (e) 'p', where 'p' stands for 'All mice are mortal'.
   (f) '(p ⊃ (q ⊃ r))', where 'p' stands for 'The next train is on time', 'q' for 'I miss my train', and 's' for 'I can arrive only five minutes late'.
   (g) 'p', where 'p' stands for 'Sam isn't over five feet tall, unless he has grown'.

Chapter III

1.

1	| q	hyp
2	| | p	hyp
3	| | | p	hyp
4	| | (p ⊃ p)	3, imp int
5	| (p ⊃ (p ⊃ p))	2–4, imp int

(c)

1	| ((p ⊃ (q ⊃ p)) ⊃ r)	hyp
2	| | p	hyp
3	| | | q	hyp
4	| | | p	2, reit
5	| | (q ⊃ p)	3–4, imp int
6	| (p ⊃ (q ⊃ p))	2–5, imp int
7	| r	1, 6, m p

(e)